Topologies on Products

Let $\{X_i\}_{i\in J}$ be a collection of topological spaces indexed by the set J.

The Cartesian product of this collection is

$$X = \prod X_i := \{(a_i)_{i\in J} \mid a_i \in X_i\}$$

i.e. it's the set of maps $x : J \to \bigcup X_i$ s.t. $x(i) \in A_i$.

How do we define a topology on X? Two natural options:

Naive approach: Box topology

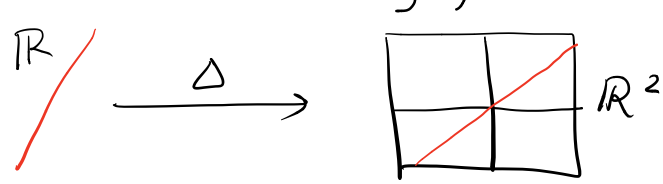
The box topology on X has basis

$$\{\prod U_i \mid U_i \subseteq X_i \text{ open}\} \quad \text{(check this is a basis)}$$

This is a natural definition, but it has weird properties:

Ex: Consider the diagonal map $\Delta : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ $(= R_0 \times R_1 \times R_2 \times \ldots)$ (or $\mathbb{R}^\omega$) defined $D(x) = (x, x, x, \ldots)$.

For finite products using the product topology $\Delta : \mathbb{R} \to \mathbb{R}^n$ is continuous (in fact it's an embedding!)



However, giving $\mathbb{R}^{\mathbb{N}}$ the box topology, $\Delta : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ is not continuous!

Consider $B = (-1,1) \times (-\frac{1}{2},\frac{1}{2}) \times (\frac{-1}{3},\frac{1}{3}) \times \dots$.

$$\Delta^{-1}(B) = \bigcap \left(-\tfrac{1}{n},\tfrac{1}{n}\right) = \{0\}, \text{ which is not open in } \mathbb{R}.$$

Better idea: Product topology

The <u>product topology</u> on $X = \prod X_i$ has basis

$$\left\{ \prod U_i \;\middle|\; \begin{array}{l} U_i \subseteq X_i \text{ is open and } U_i = X_i \\ \text{for all but finitely many } i \end{array} \right\}$$ (Check this is a basis!)

(Notice that the box and product topologies are the same if the indexing set is finite.)

Unless I specify, $\prod X_i$ will be given the product topology from now on

Thm: $f : Z \longrightarrow \prod X_i$ is continuous $\iff$ each $f_i : Z \to X_i$ is continuous.

Pf: Suppose f is continuous. $f_i = p_i \circ f$ where p_i is the ith projection.

p_i is continuous since if $U_i \subseteq X_i$ is open, $p_i^{-1}(U_i)$ is the product where the jth factor (other than $j = i$) is X_j.

Other direction: exercise. $\square$

Ex: This implies the diagonal $\Delta : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ is continuous, since Δ_i is the identity.

There is another interesting topology that sits between the box and product topologies:

The uniform topology

First we need to define a new metric on $\mathbb{R}$:

Ex: On $\mathbb{R}$, let d be the standard (Euclidean) metric.

Define $\bar{d}(x,y) = \min\{d(x,y), 1\}$

Basis for the corresponding topology consists of balls of radius ≤ 1 and all of $\mathbb{R}$.

Claim: this induces the standard topology.

If $\bar{B}_r(x)$ is a ball in the metric $\bar{d}$, then

If $r \leq 1$, $\bar{B}_r(x) = \underbrace{B_r(x)}_{\text{r-ball in standard metric}}$

If $r > 1$, $\bar{B}_r(x) = \mathbb{R} \supseteq B_r(x)$.

For $B_r(x)$ in the standard metric,

If $r \leq 1$, $B_r(x) = \bar{B}_r(x)$

If $r > 1$, $B_r(x) \supseteq B_\varepsilon(y)$ for $\forall\, y \in B_r(x)$ and some $\varepsilon \leq 1$

$B_\varepsilon(y) = \bar{B}_\varepsilon(y)$

So each topology is finer than the other, so they are the same. This is true more generally.

Def: Let (X,d) be a metric space.

$\bar{d}(x,y) = \min\{d(x,y), 1\}$ is called the <u>standard bounded metric</u> and it induces the same topology as d. (by same proof as above)

Let J be an index set. Let $X = \mathbb{R}^J$ (i.e. $\prod_{i \in J} \mathbb{R}$)

Define a metric $\bar{\rho}$ on X by:

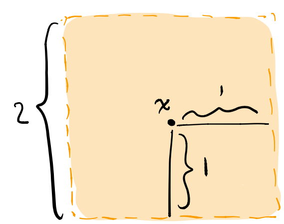
$$\bar{\rho}(\vec{x}, \vec{y}) = \sup\{\bar{d}(x_i, y_i) \mid i \in J\}$$

(Can you see why we need $\bar{d}$ here?)

where $\bar{d}$ is the standard bounded metric on $\mathbb{R}$.

This is called the <u>uniform metric</u> and it induces the <u>uniform topology</u>.

If J is finite, open balls look like square regions w/ side lengths at most 2 (and the whole space).



What if J is infinite?

Ex: Consider $U = (-1,1) \times (-1,1) \times \cdots \subseteq \mathbb{R}^{\mathbb{N}}$ w/ uniform topology. Is this the 1-ball around $(0,0,0,\ldots)$?

Let $\vec{x} = (\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots)$. Then

$\bar{\rho}(\vec{x}, \vec{0}) = 1$, so U is not in the 1-ball around $\vec{0}$!

In fact, $\mathcal{U}$ is not open! There's no ε-nhd around $\vec{x}$

Since there's always some coordinate w/in ε of 1.

What are the ε balls around $(0,0,\dots)$? If $\varepsilon=1$, it's everything.

If $\varepsilon<1$, Let $U_\delta = (-\delta,\delta)\times(-\delta,\delta)\dots$

If $\delta<\varepsilon$, then U_δ is contained in the ε-ball.

If $\vec{x}$ is within the ε ball, then $\sup\{x_i\}_{i\in\mathbb{N}} = \varepsilon' < \varepsilon$, so

$\vec{x} \in U_{\frac{\varepsilon+\varepsilon'}{2}}$.

Thus, the ε-ball is $\bigcup_{0<\delta<\varepsilon} U_\delta$

Theorem: The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology.

Pf: Let $\vec{x}\in\mathbb{R}^J$, and $\prod U_i$ be a basis element in the product topology containing $\vec{x} = (x_i)_{i\in J}$

Let $\alpha_1,\alpha_2,\dots,\alpha_n$ be the indices where $U_i \neq \mathbb{R}$.

Since U_{α_j} is open in $\mathbb{R}$, we can find an ε_j-ball (in $\bar{d}$ metric) around x_{α_j} contained in U_{α_j}. Let $\varepsilon = \min\{\varepsilon_1,\dots,\varepsilon_n\}$.

The ε-ball centered at $\vec{x}$ in the $\bar{\rho}$-metric is in $\prod U_i$.
Thus the uniform topology is finer that the product topology.

Now let B be an ε-ball centered at $\vec{x}$ in the $\bar{\rho}$ metric.

Then $\forall\ \vec{y} \in B$, we can find $\delta < \varepsilon$ s.t. $\vec{y} \in \prod_{i \in J} (x_i - \delta, x_i + \delta) \subseteq B$

so the box topology is finer than the uniform topology. $\square$

So $\mathcal{T}_{prod} \subseteq \mathcal{T}_{unif} \subseteq \mathcal{T}_{box}$

Note: Since the box and product topologies are the same if J is finite, they are all the same in this case.

However, if J is infinite, they are all different (as we saw above).